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WHEN DO TWO BANACH SPACES HAVE ISOMETRICALLY ISOMORPHIC NONSTANDARD HULLS?†

BY

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ABSTRACT

The answer to the title question is given in terms of the elementary properties of Banach spaces regarded as structures for a certain first-order language. The same question for Banach space ultrapowers is also considered. The connection between nonstandard hulls and Banach space ultrapowers derives in part from the following fact, of independent interest in nonstandard analysis: for each cardinal number κ there exist ultrapower enlargements which are κ-saturated and which have the κ-isomorphism property.

The basic first-order language L used here has as its nonlogical symbols one binary function symbol $+$ and two unary predicate symbols P and Q. We regard a Banach space E (with norm ρ) as an L-structure by taking $+_E$ to be the vector addition on E and by setting

$$P_E = \{x \mid \rho(x) \leqq 1\}$$

$$Q_E = \{x \mid \rho(x) \geqq 1\}.$$

The criterion for the existence of isometrically isomorphic nonstandard hulls is given as follows:

THEOREM 1. *The following conditions are equivalent for Banach spaces E, F over the scalar field of real numbers*:

(i) *E and F have isometrically isomorphic nonstandard hulls.*

(ii) *There is a Banach space H such that any positive sentence of L which is true in E or in F is also true in H.*

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Moreover, if E and F satisfy condition (ii) and their nonstandard hulls are constructed using any $\aleph_1$-saturated enlargement which has the $\aleph_0$-isomorphism property [3], then those two nonstandard hulls must be isometrically isomorphic (see Theorem 2).

An important fact, which is used in the proof of Theorem 1, is that any positive sentence true in E must be true in every nonstandard hull of E. This is somewhat surprising, since each nonstandard hull of E is constructed from an elementary extension of E by first taking a substructure and only then taking a homomorphic image.

We also prove the analogue of Theorem 1 for Banach space ultrapowers (in the sense of [8]). In part, our proof depends on the following result which is of technical interest itself and which provides for a connection between nonstandard hulls and Banach space ultrapowers: For each set theoretical structure $\mathcal{M}$ of nonstandard analysis and each cardinal number κ, there is an *ultrapower* enlargement of $\mathcal{M}$ with the κ-isomorphism property [3]. This fact is a straightforward consequence of the difficult ultrafilter construction given by S. Shelah in [13]. An interesting consequence of our discussion of Banach space ultraproducts is a Compactness Theorem for sets of positive formulas in L and Banach space models (Theorem 5). It follows from this result that E and F satisfy condition (ii) of Theorem 1 if for each positive sentence σ_1 true in E and each positive sentence σ_2 true in F, there is a Banach space H in which both σ_1 and σ_2 are true.

While the results in this paper were motivated by a question in nonstandard analysis, they suggest the fruitfulness of a general study of Banach spaces as *topological structures*, as part of the emerging subject of topological model theory which was initiated by Abraham Robinson in [12]. Some results in this direction will be presented in [4]. Also, Krivine and Stern have treated Banach spaces as structures for certain special formal languages in [9] [15], with emphasis on the application of model theoretic methods within Banach space theory.

1. Preliminaries

The notation and ideas in logic or model theory which are used here can be found in [1] and in [14]. The basic facts from nonstandard analysis are contained in [10] and in [11]. Recall that a formula of the language L is *positive* if it is built up from atomic formulas using only conjunction, disjunction and universal or existential quantifiers. The most important property of positive formulas is that their truth is preserved under surjective homomorphisms.

Moreover, Lyndon's Theorem asserts that if $\mathscr{A}$ and $\mathscr{B}$ are structures for L and if every positive sentence true in $\mathscr{A}$ is true in $\mathscr{B}$, then there exist elementary extensions $\mathscr{A}'$ of $\mathscr{A}$ and $\mathscr{B}'$ of $\mathscr{B}$ such that $\mathscr{B}'$ is a homomorphic image of $\mathscr{A}'$. If E is any Banach space, we will write Pos (E) for the set of positive sentences of L which are true in E.

If E is a Banach space, (with norm ρ) then the nonstandard hulls of E are constructed in the following way: let $\mathscr{M}$ be a set-theoretical structure which contains E and let $^*\mathscr{M}$ be any $\aleph_1$-saturated enlargement of $\mathscr{M}$. An element p of $^*E$ is said to be finite if $^*\rho(p)$ is finite; p is *infinitesimal* if $^*\rho(p)$ is infinitesimal. The set of finite elements of $^*E$ is denoted by fin$(^*E)$, the set of infinitesimal elements by $\mu(0)$. The nonstandard hull $\hat{E}$ is then defined to be the quotient space fin$(^*E)/\mu(0)$, with quotient map π : fin$(^*E) \to \hat{E}$. The norm $\hat{\rho}$ on $\hat{E}$ is defined by letting $\hat{\rho}(x)$ be the standard part of $^*\rho(p)$, where p is any element of fin$(^*E)$ which satisfies $\pi(p) = x$.

Nonstandard hulls were first introduced by Luxemberg [10] and have been studied by the author and L. C. Moore, Jr. [3] [5] [6] [7]. They arise naturally in many parts of nonstandard analysis. The question considered in this paper was suggested by the the rather surprising examples of isometries between nonstandard hulls which were given by the author in [3]. These examples arise when the enlargement $^*\mathscr{M}$ has the $\aleph_0$-isomorphism property, which was introduced in [3] and which will be of importance in this paper. Recall that $^*\mathscr{M}$ has the $\aleph_0$-isomorphism property (as an enlargement of $\mathscr{M}$) if the following condition holds: if $\mathscr{A}$ and $\mathscr{B}$ are elementarily equivalent structures with a finite number of relations and functions and if the domains, relations and functions of $\mathscr{A}$ and $\mathscr{B}$ are all internal objects in $^*\mathscr{M}$, then $\mathscr{A}$ and $\mathscr{B}$ are isomorphic.

Throughout this paper we let T denote the first order theory with language L whose axioms are the sentences of L which are true in every non-trivial Banach space. It is easy to check that if $\mathscr{A}$ is any model of T, then $(|\mathscr{A}|, +_{\mathscr{A}})$ is a torsion-free, divisible abelian group. Therefore it is a vector space over the field Q of rational numbers in a canonical way. Moreover, the set $P_{\mathscr{A}}$ is absolutely convex over Q; that is, if $a, b \in |\mathscr{A}|$ and $r, s \in Q$ satisfy $|r| + |s| \leqq 1$, then

$$ra +_{\mathscr{A}} sb \in P_{\mathscr{A}}.$$

(For example, the following sentence is a theorem of T:

$$\forall x \, \forall y \, \forall z \, \forall u \, \forall v \, [(Px \wedge Py \wedge x = u + u \wedge y = v + v \wedge z = u + v) \to Pz].$$

This implies that $\frac{1}{2}a +_{\mathscr{A}} \frac{1}{2}b \in P_{\mathscr{A}}$ whenever a and b are in $P_{\mathscr{A}}$.) We say that an element a of $|\mathscr{A}|$ is *finite* if there is a positive integer n such that $(1/n)a$ is in $P_{\mathscr{A}}$. The convexity property of $P_{\mathscr{A}}$ insures that the set of finite elements of $|\mathscr{A}|$ forms a vector subspace of $(|\mathscr{A}|, +_{\mathscr{A}})$ over Q. In particular the set of finite elements of $|\mathscr{A}|$ is the domain of a substructure of $\mathscr{A}$ which we denote by $\mathrm{fin}(\mathscr{A})$.

LEMMA 1. *If $\mathscr{A}$ is a model of T, then* $\mathrm{fin}(\mathscr{A})$ *is an elementary submodel of $\mathscr{A}$.*

PROOF. Let $\mathscr{A}$ be a model of T and let A_f be the set of finite elements of $\mathscr{A}$. Let κ be a cardinal number which is greater than the cardinality of $\mathscr{A}$; in particular κ is uncountable.

The fact that $\mathscr{A}$ is a model of T insures that for each positive integer n there exist $x_1, \cdots, x_n$ in A_f such that

$$1 \leqq i < j \leqq n \quad \text{implies} \quad \frac{1}{n}(x_i - x_j) \notin P_{\mathscr{A}}.$$

(Let $x_1, \cdots, x_n$ be appropriate multiples of an element x of $P_{\mathscr{A}}$ which satisfies $2x \notin P_{\mathscr{A}}$.) Using the upward Löwenheim-Skolem Theorem, there is an extension $\mathscr{B}$ of $\mathscr{A}$ and subsets B and X of $|\mathscr{B}|$ such that

(a) $(\mathscr{B}, B)$ is an elementary extension of $(\mathscr{A}, A_f)$

(b) X, B and $|\mathscr{B}|$ have cardinality κ

(c) $X \subseteq B$ and if x, y are distinct elements of X, then $\frac{1}{n}(x - y) \notin P_{\mathscr{B}}$ for every positive integer n.

Condition (a) insures that $\mathscr{B}$ is a model of T and that B is a vector subspace of $(|\mathscr{B}|, +_{\mathscr{B}})$ over Q. Let $\mathscr{B}'$ be the substructure of $\mathscr{B}$ whose domain is B. Condition (a) implies that $\mathscr{B}'$ is an elementary extension of $\mathrm{fin}(\mathscr{A})$. Let B_f be the set of finite elements of $|\mathscr{B}|$. Condition (a) insures that $B_f \subseteq B$ and conditions (b) and (c) insure that the quotient vector space B/B_f has cardinality κ. Therefore B/B_f and $|\mathscr{B}|/B_f$ both have dimension κ as vector spaces over Q. It follows that there is a vector space isomorphism f from B onto $|\mathscr{B}|$ which is the identity when restricted to B_f. But $P_{\mathscr{B}}$ is contained in B_f and $|\mathscr{B}| \sim B_f$ is contained in $Q_{\mathscr{B}}$. Therefore f is an isomorphism of $\mathscr{B}'$ onto $\mathscr{B}$.

That is, $\mathrm{fin}(\mathscr{A})$ and $\mathscr{A}$ have elementary extensions which are isomorphic by a mapping which is the identity on $|\mathrm{fin}(\mathscr{A})|$. It follows that $\mathrm{fin}(\mathscr{A})$ is an elementary submodel of $\mathscr{A}$, completing the proof.

In many ways Lemma 1 is the key to the results which follow. The analogous statement for stronger languages than L, such as those used in [9] [15], can be shown to be false. For such languages the truth of positive formulas is not necessarily preserved on passing from E to a nonstandard hull or Banach space ultrapower of E.

2. Nonstandard hulls

PROPOSITION 1. *If $\hat{E}$ is a nonstandard hull of the Banach space E, then* $\mathrm{Pos}(E) \subseteq \mathrm{Pos}(\hat{E})$.

PROOF. Let $^*\mathcal{M}$ be the $\aleph_1$-saturated enlargement used in constructing $\hat{E}$. If $\mathcal{A} = (E, +_E, P_E, Q_E)$ is the L-structure associated with E, then $^*\mathcal{A} = (^*E, ^*+_E, ^*P_E, ^*Q_E)$ is an elementary extension of $\mathcal{A}$, by the transfer principle. Noting that $^*P_E = \{p \in {^*E} \mid {^*\rho}(p) \leqq 1\}$, we see that $^*\rho(q)$ is finite if and only if there is a (standard) positive integer n with $(1/n)q \in {^*P_E}$. Therefore, Lemma 1 implies that the substructure of $^*\mathcal{A}$ with domain $\mathrm{fin}(^*E)$ is an elementary substructure of $^*\mathcal{A}$ — indeed, that substructure is just $\mathrm{fin}(^*\mathcal{A})$. Since $E \subseteq \mathrm{fin}(^*E)$, it follows that $\mathrm{fin}(^*\mathcal{A})$ is an elementary extension of $\mathcal{A}$. Now the mapping $\pi: \mathrm{fin}(^*E) \to \hat{E}$ is easily seen to be a homomorphism of $\mathrm{fin}(^*\mathcal{A})$ onto $\hat{E}$. The desired result follows immediately from the fact that truth of positive sentences is preserved under surjective homomorphisms.

PROPOSITION 2. *Let E, F be Banach spaces in $\mathcal{M}$ and let $^*\mathcal{M}$ be an $\aleph_1$-saturated enlargement of $\mathcal{M}$ which has the $\aleph_0$-isomorphism property. If* $\mathrm{Pos}(E) \subseteq \mathrm{Pos}(F)$, *then the nonstandard hulls of E and F constructed using $^*\mathcal{M}$ are isometrically isomorphic.*

PROOF. Since $\mathrm{Pos}(E) \subseteq \mathrm{Pos}(F)$, Lyndon's Theorem implies that there exist L-structures $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{A}$ is an elementary extension of E, $\mathcal{B}$ is an elementary extension of F and there is a homomorphism f of $\mathcal{A}$ onto $\mathcal{B}$. It may be assumed that the cardinality of $\mathcal{A}$ and $\mathcal{B}$ is $\max(\mathrm{card}\, E, \mathrm{card}\, F)$ and therefore we may take $\mathcal{A}, \mathcal{B}$ and f to be elements of $\mathcal{M}$.

Now the transfer principle insures that $^*f$ is a homomorphism of $^*\mathcal{A}$ onto $^*\mathcal{B}$. Since $^*\mathcal{A}$ is elementarily equivalent to E, and hence to $(^*E, ^*+_E, ^*P_E, ^*Q_E) = \mathcal{E}$, the fact that $^*\mathcal{M}$ has the $\aleph_0$-isomorphism property insures that $^*\mathcal{A}$ and $\mathcal{E}$ are isomorphic. Similarly $^*\mathcal{B}$ is isomorphic to $\mathcal{F} = (^*F, ^*+_F, ^*P_F, ^*Q_F)$. Therefore $^*f$ induces a homomorphism g of $\mathcal{E}$ onto $\mathcal{F}$. The fact that g maps $^*P_E$ into $^*P_F$ and preserves addition implies that g maps finite elements of $^*E$ to finite elements of $^*F$ and maps infinitesimals to infinitesimals. If $p \in {^*E}$ is infinite, so that $(1/n)p \in {^*Q_E}$ for all standard n, then $(1/n)g(p) \in {^*Q_F}$ for all such n. Therefore, $g(p)$ is infinite in $^*F$ whenever p is infinite in $^*E$. A similar argument shows that $g(p)$ is infinitesimal only when p is infinitesimal. We may therefore define $\hat{g}$ from $\hat{E}$ onto $\hat{F}$ by

$$\hat{g}(x) = \pi_F(g(p)) \quad \text{where} \quad \pi_E(p) = x.$$

It is immediate that $\hat{g}$ is a bijection and preserves addition and that $\hat{g}$ maps the unit ball of $\hat{E}$ onto the unit ball of $\hat{F}$. A simple argument shows that $\hat{g}$ is a linear isometry, completing the proof.

THEOREM 2. *Let* $^*\mathcal{M}$ *be an* $\aleph_1$*-saturated enlargement of* $\mathcal{M}$ *with the* $\aleph_0$*-isomorphism property. For any two Banach spaces* E, F *in* $\mathcal{M}$ *the following conditions are equivalent, where* $\hat{E}$ *and* $\hat{F}$ *are the nonstandard hulls constructed using* $^*\mathcal{M}$:

(i) $\hat{E}$ *and* $\hat{F}$ *are isometrically isomorphic.*

(ii) $\mathrm{Pos}(\hat{E}) = \mathrm{Pos}(\hat{F})$.

(iii) *For some Banach space* H, $\mathrm{Pos}(\hat{E}) \cup \mathrm{Pos}(\hat{F}) \subseteq \mathrm{Pos}(H)$.

(iv) *For some Banach space* H, $\mathrm{Pos}(E) \cup \mathrm{Pos}(F) \subseteq \mathrm{Pos}(H)$.

PROOF. The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are obvious. Proposition 1 yields (iii) $\Rightarrow$ (iv) immediately.

Now suppose that E, F, H satisfy condition (iv). If we knew that H were in $\mathcal{M}$, then Proposition 2 would imply that $\hat{E}$ and $\hat{F}$ were each isometrically isomorphic to $\hat{H}$ and hence to each other. If H is not in $\mathcal{M}$, it suffices to show that there is a Banach space H' in $\mathcal{M}$ which satisfies $\mathrm{Pos}(H') = \mathrm{Pos}(H)$. Moreover, since $\mathcal{M}$ contains sets of cardinality $2^{\aleph_0}$, we need only show that there is a Banach space H' of cardinality $2^{\aleph_0}$ with $\mathrm{Pos}(H') = \mathrm{Pos}(H)$. Actually such an H' exists which is an elementary subspace of H (relative to formulas in L). To obtain such an H', construct inductively a closed subspace H_α of H for each countable ordinal α such that

(a) H_α has cardinality $2^{\aleph_0}$ for each α

(b) $H_\alpha \subseteq H_\beta$ whenever $\alpha \leqq \beta$

(c) for each α there is an elementary substructure $\mathcal{A}_\alpha$ of H (for the language L) such that

$$H_\alpha \subseteq |\mathcal{A}_\alpha| \subseteq H_{\alpha+1}.$$

(To construct $H_{\alpha+1}$, let $\mathcal{A}_\alpha$ be an elementary substructure of H such that $|\mathcal{A}_\alpha|$ contains H_α and has cardinality $2^{\aleph_0}$. Let $H_{\alpha+1}$ be the closure of $|\mathcal{A}_\alpha|$. At limit ordinals β, let H_β be the closure of $\bigcup\{H_\alpha \mid \alpha < \beta\}$.)

Letting H' be the union of H_α for all countable α will produce the desired result: any sequence in H' is contained in some H_α, so H' is a closed subspace of H; evidently H' has cardinality $2^{\aleph_0}$; H' is an elementary substructure of H since H' is the union of the chain of structures $\mathcal{A}_\alpha$ described in (c). This completes the proof.

We note here that the proof of Theorem 1 is now complete: If $\hat{E}$ is a nonstandard hull of E and if $\hat{F}$ is a nonstandard hull of F (possibly constructed using *different* $\aleph_1$-saturated enlargements) and if $\hat{E}$ and $\hat{F}$ are isometrically isomorphic, then

$$\mathrm{Pos}(E) \subseteq \mathrm{Pos}(\hat{E}) = \mathrm{Pos}(\hat{F}) \supseteq \mathrm{Pos}(F)$$

by Proposition 1. Therefore E, F satisfy condition (ii) of Theorem 1. Conversely, suppose E, F satisfy that condition. Take $^*\mathcal{M}$ to be an $\aleph_1$-saturated enlargement of an $\mathcal{M}$ containing E and F, such that $^*\mathcal{M}$ has the $\aleph_0$-isomorphism property ($^*\mathcal{M}$ exists by [3, theor. 1.3].) Then the nonstandard hulls of E, F constructed using $^*\mathcal{M}$ are isometrically isomorphic, by Theorem 2.

COROLLARY 1. *Let $^*\mathcal{M}$ be an $\aleph_1$-saturated enlargement of $\mathcal{M}$ which has the $\aleph_0$-isomorphism property. If E is a Banach space in $\mathcal{M}$, then* $\mathrm{Pos}(\hat{E})$ *is the largest set S of positive sentences of L such that* $\mathrm{Pos}(E) \subseteq S$ *and for some Banach space H, $S \subset$* $\mathrm{Pos}(H)$. *In particular,* $\mathrm{Pos}(\hat{E})$ *depends only on* $\mathrm{Pos}(E)$ *and not on $\mathcal{M}$ or $^*\mathcal{M}$.*

PROOF. The set $\mathrm{Pos}(\hat{E})$ satisfies the stated condition since $\hat{E}$ is a Banach space and $\mathrm{Pos}(E) \subseteq \mathrm{Pos}(\hat{E})$ by Proposition 2. Conversely, say S is a positive set of sentences and for some Banach space H $\mathrm{Pos}(E) \subseteq S \subseteq \mathrm{Pos}(H)$. By the argument given in the proof of Theorem 2 we may assume that H is in $\mathcal{M}$. By Proposition 2, $\hat{E}$ and $\hat{H}$ are isometrically isomorphic so that

$$S \subseteq \mathrm{Pos}(H) \subseteq \mathrm{Pos}(\hat{H}) = \mathrm{Pos}(\hat{E})$$

as claimed.

REMARK. Call a set S of positive sentences of L *maximal* if there is a Banach space E such that $\mathrm{Pos}(E) = S$ and and for each Banach space H with $\mathrm{Pos}(H) \supseteq S$ we actually have $\mathrm{Pos}(H) = S$. Corollary 1 asserts that if $\hat{E}$ is a nonstandard hull constructed using an $\aleph_1$-saturated enlargement with the $\aleph_0$-isomorphism property, then $\mathrm{Pos}(\hat{E})$ is a maximal set of positive sentences.

Moreover, suppose $^*\mathcal{M}$ is a fixed $\aleph_1$-saturated enlargement with the $\aleph_0$-isomorphism property, of some fixed structure $\mathcal{M}$. By the argument given in the proof of Theorem 2, for each Banach space H there is a Banach space H' in $\mathcal{M}$ with $\mathrm{Pos}(H) = \mathrm{Pos}(H')$. In particular, if $\mathrm{Pos}(H)$ is maximal, then

$$\mathrm{Pos}(\hat{H}') = \mathrm{Pos}(H') = \mathrm{Pos}(H)$$

by Corollary 1. Therefore each maximal set S of positive sentences is realized as $\mathrm{Pos}(\hat{E})$ for some nonstandard hull constructed using the fixed $^*\mathcal{M}$. Using this and Theorem 2 we see that there is a 1–1 correspondence between isometric isomorphism types of nonstandard hulls constructed using $^*\mathcal{M}$ and maximal sets of positive sentences in L.

EXAMPLE. There is a Banach space E such that $\operatorname{Pos}(E)$ is not maximal. Namely, let $\{r_n\}$ be a sequence of rational numbers decreasing (strictly) to 1 and let E be the l_2 sum of the family $\{l_{r_n}(2)\}$ of 2-dimensional spaces. (See [2, p. 35]). Since l_2 and each $l_{r_n}(2)$ are strictly convex, so is E. In particular, no 2-dimensional subspace of E is isometrically isomorphic to $l_1(2)$. However it is easy to see that any nonstandard hull $\hat{E}$ of E must contain a subspace which is isometrically isomorphic to $l_1(2)$.

Now there is a positive sentence σ with the property that σ is true in the Banach space H if and only if H contains a 2-dimensional subspace which is isometrically isomorphic to $l_1(2)$. Namely, σ may be taken to be

$$\exists x\, \exists y\, (Px \wedge Qx \wedge Py \wedge Qy \wedge P(x+y) \wedge Q(x+y) \wedge \exists z\, (x = y + z \wedge Pz \wedge Qz)).$$

Then σ asserts the existence of x and y of norm equal to 1 such that both $x+y$ and $x-y$ have norm 1.

For the space E constructed above, $\sigma \notin \operatorname{Pos}(E)$ but $\sigma \in \operatorname{Pos}(\hat{E})$ for any nonstandard hull $\hat{E}$ of E. It follows that $\operatorname{Pos}(E)$ is not maximal.

3. Banach space ultraproducts

In [8] an ultraproduct construction for Banach spaces was introduced; the construction can be described as follows. Given a family $\{E_i \mid i \in I\}$ of Banach spaces and an ultrafilter D on I, let $\mathcal{A}$ be the ordinary ultraproduct D-prod $\langle E_i \mid i \in I\rangle$ of the Banach spaces E_i as structures for the language L. Note that $\mathcal{A}$ is a model of T. Call an element a of $|\mathcal{A}|$ *infinitesimal* if $(1/m)a \in P_{\mathcal{A}}$ for every positive integer m. The absolute convexity of $P_{\mathcal{A}}$ over Q insures that the infinitesimal elements of $|\mathcal{A}|$ form a vector subspace of $(|\mathcal{A}|, +_{\mathcal{A}})$. Let E be the quotient space of $\operatorname{fin}(\mathcal{A})$ by the subspace of infinitesimal elements, and let $\nu : \operatorname{fin}(\mathcal{A}) \to E$ be the quotient mapping. It turns out that there is a norm ρ on E with

$$\nu(P_{\mathcal{A}}) = \{x \mid \rho(x) \leqq 1\}$$

and

$$\nu(Q_{\mathcal{A}} \cap |\operatorname{fin} \mathcal{A}|) = \{x \mid \rho(x) \geqq 1\}$$

and under which E becomes a Banach space (as long as D is countably incomplete). This Banach space E is the *Banach space ultraproduct* of the family $\{E_i \mid i \in I\}$ by the ultrafilter D. If $E_i = F$ for every $i \in I$, then E is called a *Banach space ultrapower* of F. Since E is a homomorphic image of $\text{fin}(\mathscr{A})$ under the mapping ν, any positive sentence true in $\mathscr{A}$ is true in E by Lemma 1. From this fact and the basic property of ordinary ultraproducts in model theory, the following result is immediate.

PROPOSITION 3. *Let E be the Banach space ultraproduct of the family $\{E_i \mid i \in I\}$ of Banach spaces by the ultrafilter D and let σ be a positive sentence of L. If $\{i \in I \mid \sigma$ is true in $E_i\} \in D$, then σ is true in E.*

In particular if E is a Banach space ultrapower of F, then $\text{Pos}(F) \subseteq \text{Pos}(E)$.

THEOREM 3. *For each set-theoretical structure $\mathcal{M}$ of nonstandard analysis and each cardinal number κ there is an ultrapower enlargement of $\mathcal{M}$ which has the κ-isomorphism property.*

PROOF. We may assume that κ is greater than the cardinality of any set in $\mathcal{M}$. In that case any ultrapower extension of $\mathcal{M}$ which has the κ-isomorphism property will automatically be an enlargement, since it is κ-saturated by the argument given in [3, theor. 1.5].

Shelah [13] has shown that there exists an ultrafilter D on a set I which satisfies the following condition: if $\langle \mathscr{A}_i \mid i \in I\rangle$ and $\langle \mathscr{B}_i \mid i \in I\rangle$ are families of structures for the same first order language L_1 and if the cardinalities of each $|\mathscr{A}_i|$, each $|\mathscr{B}_i|$ and the set of non-logical symbols of L_1 are all less than κ, then the ultraproducts D-prod $\langle \mathscr{A}_i \mid i \in I\rangle$ and D-prod $\langle \mathscr{B}_i \mid i \in I\rangle$ are isomorphic whenever they are elementarily equivalent. (Shelah does not specify that the cardinality of L_1 must be bounded, but this requirement is necessary.) Let $^*\mathcal{M}$ be the ultrapower extension of $\mathcal{M}$ constructed using the ultrafilter D (see [10] for this construction).

Now assume that $\mathscr{A}$ and $\mathscr{B}$ are elementarily equivalent structures for a language L_1 which has fewer than κ non-logical symbols and assume that the domains, functions and relations of $\mathscr{A}$ and $\mathscr{B}$ are all internal objects in the extension $^*\mathcal{M}$ of $\mathcal{M}$. It must be shown that $\mathscr{A}$ and $\mathscr{B}$ are isomorphic. Let A and B be sets in $\mathcal{M}$ such that the internal sets $|\mathscr{A}|$ and $|\mathscr{B}|$ satisfy $|\mathscr{A}| \subseteq {}^*A$ and $|\mathscr{B}| \subseteq {}^*B$. Since $|\mathscr{A}|$ is internal there exists a function $f: I \to \{X \mid X \subseteq A\}$ such that $|\mathscr{A}|$ corresponds (via the construction of $^*\mathcal{M}$) to the ultraproduct element f/D. (Here f/D is the D-equivalence class of functions from I into $\mathcal{M}$ which contains f.) Similarly there is a function $g: I \to \{Y \mid Y \subseteq B\}$ such that g/D corresponds to the internal set $|\mathscr{B}|$. Given a non-logical symbol s of L_1 there are functions f_s and g_s from I into $\mathcal{M}$ such that the internal objects $s_{\mathscr{A}}$ and $s_{\mathscr{B}}$ correspond to f_s/D and g_s/D respectively. Moreover, if s is (for example) an

n-place predicate symbol, then f_s and g_s may be chosen so that, for each $i \in I$, $f_s(i)$ is an n-ary relation on $f(i)$ and $g_s(i)$ is an n-ary relation on $g(i)$. Similar assumptions may be made in respect to the function symbols of L_1.

Now let $\mathscr{A}_i$ be the L_1-structure whose domain is $f(i)$ and such that for each non-logical symbol s of L_1, $s_{\mathscr{A}_i} = f_s(i)$. Let $\mathscr{B}_i$ be the structure with domain $g(i)$ and such that $s_{\mathscr{B}_i} = g_s(i)$ for each symbol s. It follows that $\mathscr{A}$ is isomorphic to the ultraproduct D-prod $\langle \mathscr{A}_i \mid i \in I \rangle$ and $\mathscr{B}$ is isomorphic to D-prod$\langle \mathscr{B}_i \mid i \in I \rangle$. Since the domain of each $\mathscr{A}_i$ or $\mathscr{B}_i$ is a set in $\mathscr{M}$, it has cardinality less than κ. Therefore, the basic property of the ultrafilter D insures that $\mathscr{A}$ and $\mathscr{B}$ are isomorphic, completing the proof.

Suppose now that E is a Banach space in $\mathscr{M}$ and that ${}^*\mathscr{M}$ is the ultrapower extension of $\mathscr{M}$ constructed using an ultrafilter D. Then ${}^*E$ corresponds to the ultrapower of E by D. Therefore the nonstandard hull of E constructed using this ${}^*\mathscr{M}$ is isometrically isomorphic to the Banach space ultrapower of E by the ultrafilter D. This connection between nonstandard hulls and Banach space ultrapowers leads to the following analogue of Theorem 1.

THEOREM 4. *The following conditions are equivalent for Banach spaces E, F over the scalar field of real numbers*:

(i) *E and F have isometrically isomorphic Banach space ultrapowers.*

(ii) *There is a Banach space H such that* $\mathrm{Pos}(E) \cup \mathrm{Pos}(F) \subseteq \mathrm{Pos}(H)$.

PROOF. The implication (i) $\Rightarrow$ (ii) follows from Proposition 3. If E and F satisfy (ii), let ${}^*\mathscr{M}$ be an ultrapower enlargement with the $\aleph_1$-isomorphism property of some $\mathscr{M}$ which contains E and F, obtained using Theorem 3. By [3, theor. 1.5] ${}^*\mathscr{M}$ is an $\aleph_1$-saturated enlargement of $\mathscr{M}$. By Theorem 2 the nonstandard hulls of E and F constructed using ${}^*\mathscr{M}$ are isometrically isomorphic, and these are the desired Banach space ultrapowers.

Another interesting consequence of Theorem 3 and the use of Banach space ultraproducts is the following Compactness Theorem for sets of positive sentences of L and Banach space models.

THEOREM 5. *If Σ is a set of positive sentences of L and for each finite subset Σ' of Σ there is a Banach space H with $\Sigma' \subseteq \mathrm{Pos}(H)$, then there is a Banach space H with $\Sigma \subseteq \mathrm{Pos}(H)$.*

The argument by which Theorem 5 is derived from Theorem 3 is familiar and will be omitted. It should be noted that there is no analogous Compactness Theorem for arbitrary sets of sentences in L (see [4]).

It follows from Theorem 5 that the condition

(i) *There exists a Banach space H such that* $\operatorname{Pos}(E) \cup \operatorname{Pos}(F) \subseteq \operatorname{Pos}(H)$ (which has been shown to be equivalent to the existence of isometrically isomorphic nonstandard hulls or Banach space ultrapowers of E and F) is equivalent to the condition

(ii) *For each σ_1 in* $\operatorname{Pos}(E)$ *and each σ_2 in* $\operatorname{Pos}(F)$, *there is a Banach space H such that* $\{\sigma_1, \sigma_2\} \subseteq \operatorname{Pos}(H)$.

REMARK. While the results in this paper have been restricted to real Banach spaces, they can easily be extended to analogous results for complex Banach spaces. All that is necessary is to expand the language L by adding a unary function symbol which is interpreted as the operation of scalar multiplication by $\sqrt{-1}$.

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